NASA TECHNICAL TRANSLATION NASA TT F-15,073

PERTURBATION OF A NEARLY-CIRCULAR ORBIT

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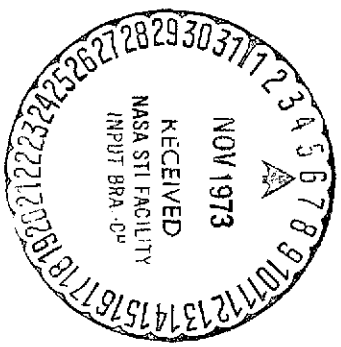
(NASA-TT-F-15073) PERTURBATION OF A NEARLY-CIRCULAR ORBIT (Linguistic Systems, Inc., Cambridge, Mass.) ~~19~~ 20 p HC CSCL 22C G3/30

N73-33825

Unclas 19875

Translation of: "Vozmushcheniya Pochti-Krugovoy Orbity," Ucheniyye Zapiski Leningradskogo Gosudarstvennogo Universiteta (Scientific Papers of the Leningrad State University) No. 363, Seriya Matematicheskikh Nauk (Mathematical Sciences Series) Issue 48, Trudy Astronomicheskoy Observatorii (Transactions of the Astronomical Observatory) Vol. 29, Leningrad, 'Leningradskogo Universitet' Press, 1973, pp. 163-176.



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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION
WASHINGTON, D.C. 20546 NOVEMBER 1973

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STANDARD TITLE PAGE

1. Report No.	2. Government Accession No.	3. Recipient's Catalog No.
NASA TT F- 15,073		

4. Title and Subtitle	5. Report Date
Perturbation of a Nearly-Circular Orbit	November 1973
	6. Performing Organization Code

7. Author(s)	8. Performing Organization Report No.
Ye. I. Timoshkova and K. V. Kholshevnikov	10. Work Unit No.

9. Performing Organization Name and Address	11. Contract or Grant No.
LINGUISTIC SYSTEMS, INC. 116 AUSTIN STREET CAMBRIDGE, MASSACHUSETTS 02139	NASW-2482
	13. Type of Report & Period Covered TRANSLATION
12. Sponsoring Agency Name and Address NATIONAL AERONAUTICS AND SPACE ADMINISTRATION WASHINGTON, D.C. 20546	14. Sponsoring Agency Code

15. Supplementary Notes

Translation of: "Vozmushcheniya Pochti-Krugovoy Orbity," Ucheniyye Zapiski Leningradskogo Gosudarstvennogo Universiteta (Scientific Papers of the Leningrad State University) No. 363, Seriya Matematicheskikh Nauk (Mathematical Sciences Series) Issue 48, Trudy Astronomicheskoy Observatorii (Transactions of the Astronomical Observatory) Vol. 29, Leningrad, Leningrad University Press, 1973, pp. 163-176.

16. Abstract

The motion of a satellite in the gravitational field of nonspherical rotating planet is considered. The motion in the field of two fixed centers is taken as the unperturbed one. The Lagrange-type equations are derived for the perturbed elements of the intermediate nearly-circular orbit. The integration of these equations is fulfilled and the expressions are given for the calculation of the secular, long-period and short-period perturbations caused by an arbitrary harmonics of the planet's potential.

17. Key Words (Selected by Author(s))	18. Distribution Statement
	UNCLASSIFIED - UNLIMITED

19. Security Classif. (of this report)	20. Security Classif. (of this page)	21. No. of Pages	22. Price
UNCLASSIFIED	UNCLASSIFIED	18	

PERTURBATION OF A NEARLY-CIRCULAR ORBIT

Ye. I. Timoshkova and K. V. Kholshevnikov

We will examine the motion of a satellite in the gravitational field of a nonspherical revolving planet. We will take the motion in the field of two fixed centers as the unperturbed one. Further in the work, . . we will call the corresponding orbits Eulerian, since the problem of two fixed centers was first reduced to quadratics by Leonard Euler. It is known [1,2], that the functions, describing the perturbations of the elements of an Eulerian orbit, suffer a discontinuity during the turning of the eccentricity e to zero. They lose their practical utility even for a sufficiently small eccentricity. What value of e is to be considered sufficiently small depends on the specific conditions of motion of the celestial body. In addition it is necessary to consider the changes in the initial value of e_o under the action of the perturbation. /163*

As was shown in [1], the amplitude of the long period perturbation in e is proportional to J_{km}/J_{20}. As an example, for a satellite close to the Earth, the value is on the order of 10^{-3}. Thus for a small initial value of e_o the eccentricity may become as small as desired, including e=0. In this case, the formulae for the perturbation of the elements obtained in [1], are inapplicable. /164

1. ELEMENTS OF THE NEARLY-CIRCULAR ORBIT. THE LAGRANGIAN TYPE EQUATIONS.

We defined the unperturbed Eulerian orbit by the six elements

*Numbers in the right-hand margin indicate pagination in the foreign text.

$$a,\ e,\ s,\ M,\ g,\ \Omega, \tag{1.1}$$

of which the first three elements are constant; the elements M, g, and Ω are linear functions of time

$$\frac{dM}{dt}=n_1,\quad \frac{dg}{dt}=n_2,\quad \frac{d\Omega}{dt}=n_3, \tag{1.2}$$

$$\begin{aligned}
&n_1=n(1+\mu_1),\ n_2=n\mu_2(1+\mu_1),\ n_3=n\mu_3(1+\mu_1),\\
&n=\varkappa a^{-\frac{3}{2}}\left\{1-\frac{3}{2}\varepsilon^2(1-e^2)(1-s^2)+\right.\\
&\left.+\frac{3}{8}\varepsilon^4(1-e^2)(1-s^2)\left[1+11s^2-e^2(1-5s^2)\right]\right\},\\
&\mu_1=-\frac{\varepsilon^4}{16}\sqrt{(1-e^2)^3}(24-96s^2+75s^4),\\
&\mu_2=\frac{\varepsilon^2}{4}(12-15s^2)+\frac{\varepsilon^4}{64}[288-1296s^2+1035s^4-\\
&-e^2(144+288s^2-510s^4)],\\
&\mu_3=-\frac{3}{2}\varepsilon^2\sqrt{1-s^2}\left[1+\frac{\varepsilon^2}{8}(6-17s^2-24e^2s^2)\right],
\end{aligned} \tag{1.3}$$

where

$$\varepsilon^2=\frac{|J_{20}|r_0^2}{a^2(1-e^2)^2}. \tag{1.4}$$

Here r_0 is the equatorial radius of the planet; J_{20} is the coefficient for the second zonal harmonic in the expansion of the gravitational potential with respect to the spherical functions. For close satellites, ε^2 is a small parameter on the order of J_{20} (for an AES* $\varepsilon^2<1/900$, for a satellite of Mars $\varepsilon^2<1/400$).

The formulae presented here and in further work are presented as expansions to the order of ε^2 to ε^4 inclusive. In the remainder the expressions are of closed form.

We will examine the orbits for which the eccentricity e is small. For the determination let $e<0.01$. We will introduce, in place of the elements e, M, and g, the elements λ, β_1 and β_2 according to the formulae

$$\lambda=M+g,\ \beta_1=e\cos g,\ \beta_2=e\sin g. \tag{1.5}$$

*TRANSLATOR'S NOTE: AES = ARTIFICIAL EARTH SATELLITE

More convenient for the furthest computations are found to be not the components of the Laplacian vectors β_1 and β_2, but the complex conjugate values γ, $\bar{\gamma}$; the Laplacian vector and its reflection relative to the line of apses /165

$$\gamma = \beta_1 + i\beta_2, \quad \bar{\gamma} = \beta_1 - i\beta_2,$$
$$\beta_1 = \frac{\gamma + \bar{\gamma}}{2}, \quad \beta_2 = \frac{\gamma - \bar{\gamma}}{2i}.$$

Thus, in the case of small initial eccentricty, $e_0 < 0.01$ we will define the intermediate orbit of the system elements

$$a, s, \lambda, \Omega, \beta_1, \beta_2 \tag{1.6}$$

or

$$a, s, \lambda, \Omega, \gamma, \bar{\gamma}, \tag{1.7}$$

which are connected with the elements of (1.1) by formulae (1.5) or

$$\lambda = M + g, \quad \gamma = e \cdot \exp(ig), \quad \bar{\gamma} = e \cdot \exp(-ig). \tag{1.8}$$

We will obtain the equations for the perturbation elements (1.7). As was shown in [1], the equations for the perturbation of osculating elements a, e, s, M, g, and Ω have the form of the equations of Lagrange for osculating Keplerian elements, if we choose terms of the order of $J_{20} \cdot J_{km}$ ((for the Earth $\sim 10^{-9}$) and add the average movements n_1, n_2, and n_3. Keeping in mind the relationship

$$\frac{\partial R}{\partial e} = \exp(ig)\frac{\partial R}{\partial \gamma} + \exp(-ig)\frac{\partial R}{\partial \bar{\gamma}}, \quad \frac{\partial R}{\partial M} = \frac{\partial R}{\partial \lambda},$$
$$\frac{\partial R}{\partial g} = i\gamma\frac{\partial R}{\partial \gamma} - i\bar{\gamma}\frac{\partial R}{\partial \bar{\gamma}} + \frac{\partial R}{\partial \lambda},$$

we finally obtain the following system of differential equations:

$$\frac{da}{dt} = \frac{2\sqrt{a}}{\varkappa}\frac{\partial R}{\partial \lambda},$$

$$\frac{ds}{dt} = \frac{1-s^2}{\varkappa s\sqrt{a(1-e^2)}}\left[\frac{\partial R}{\partial \lambda} + i\left(\gamma\frac{\partial R}{\partial \gamma} - \bar{\gamma}\frac{\partial R}{\partial \bar{\gamma}}\right)\right] - \frac{\sqrt{1-s^2}}{\varkappa s\sqrt{a(1-e^2)}}\frac{\partial R}{\partial \Omega},$$

$$\frac{d\Omega}{dt} = n_3 + \frac{\sqrt{1-s^2}}{\varkappa s\sqrt{a(1-e^2)}}\frac{\partial R}{\partial s},$$

$$\frac{d\lambda}{dt} = n_1 + n_2 + \frac{\sqrt{1-e^2}}{\varkappa\sqrt{a}(1+\sqrt{1-e^2})}\left(\gamma\frac{\partial R}{\partial \gamma} + \bar{\gamma}\frac{\partial R}{\partial \bar{\gamma}}\right) - \frac{2\sqrt{a}}{\varkappa}\frac{\partial R}{\partial a} - \frac{1-s^2}{\varkappa s\sqrt{a(1-e^2)}}\frac{\partial R}{\partial s},$$

$$\frac{d\gamma}{dt} = in_2\gamma - \frac{i(1-s^2)}{\varkappa s\sqrt{a(1-e^2)}}\gamma\frac{\partial R}{\partial s} - \frac{\sqrt{1-e^2}}{\varkappa\sqrt{a}(1+\sqrt{1-e^2})}\gamma\frac{\partial R}{\partial \lambda} + \frac{2i\sqrt{1-e^2}}{\varkappa\sqrt{a}}\frac{\partial R}{\partial \bar{\gamma}},$$

$$\frac{d\bar{\gamma}}{dt} = -in_2\bar{\gamma} + \frac{i(1-s^2)}{\varkappa s\sqrt{a(1-e^2)}}\bar{\gamma}\frac{\partial R}{\partial s} - \frac{\sqrt{1-e^2}}{\varkappa\sqrt{a}(1+\sqrt{1-e^2})}\bar{\gamma}\frac{\partial R}{\partial \lambda} - \frac{2i\sqrt{1-e^2}}{\varkappa\sqrt{a}}\frac{\partial R}{\partial \gamma}. \qquad (1.9)$$

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2. THE PERTURBATION FUNCTIONS AND THEIR PARTIAL DERIVATIVES

On the right sides of (1.9), terms of the order of $J_{20}\cdot J_{km}$ are dropped. Thus [1] the expansion of the perturbation function R according to the elements of (1.1) may be taken as Keplerian. Following [1, 3], we represent R in the form

$$R = \varkappa^2\sum_{k=2}^{\infty}\sum_{m=0}^{k}\bar{J}_{km}R_{km},$$

$$R_{km} = \frac{r_0^k}{a^{k+1}}\sum_{j=0}^{k}\sum_{q=-\infty}^{\infty}F_{kmj}(I)\,X_{k-2j-q}^{-k-1,\,k-2j}(e)\times$$
$$\times\cos\left\{(k-2j-q)M + (k-2j)g + m[\Omega - n_4(t-t_0) - w_{km}] + \left[(m-k) + 2E\left(\frac{k-m}{2}\right)\right]\frac{\pi}{2}\right\}. \qquad (2.1)$$

Here $F_{kmj}(I)$ is the function of inclination (sin I = s) in the terms of Kaul; X(e) is the Hansen coefficient; n_4 is the

angular velocity of the planet; t_0 is the time of the crossing of the spring equinox across the initial meridian. The constants J_{km} and w_{km} are connected with the distribution of mass inside the planet; $\bar{J}_{km}$ are the constant coefficients

$$\begin{gathered}\bar{J}_{20}=0,\ \bar{J}_{40}=J_{40}-J_{20}^2,\ \bar{J}_{60}=J_{60}-J_{20}^3,\\ \bar{J}_{km}=J_{km}\ (m\neq 0;\ \text{or}\ m=0,\ k\neq 2,\ 4,\ 6).\end{gathered} \tag{2.2}$$

For the gravitational field of the Earth $\bar{J}_{km}\sim 10^{-6}$. In order to integrate the system (1.9), it is necessary to express the perturbation function through the elements of (1.7). We will introduce the modified coefficients of Hansen with the aid of the relation

$$X_p^{n,m}(e)=e^{|m-p|}Y_p^{n,m}(e) \tag{2.3}$$

and define

$$\begin{aligned}D&=(k-2j-q)\lambda+m[\Omega-n_4(t-t_0)-w_{km}]+\\&+\left[m-k+2E\left(\frac{k-m}{2}\right)\right]\frac{\pi}{2},\end{aligned} \tag{2.4}$$

$$\begin{aligned}\gamma^{|q|}&=e^{|q|}\exp iqg=\begin{cases}\gamma^{|q|},\ \text{если}\ q\geqslant 0,\\ \bar{\gamma}^{|q|},\ \text{если}\ q\leqslant 0,\end{cases}\\ \bar{\gamma}^{|q|}&=e^{|q|}\exp(-iqg)=\begin{cases}\bar{\gamma}^{|q|},\ \text{если}\ q\geqslant 0,\\ \gamma^{|q|},\ \text{если}\ q\leqslant 0.\end{cases}\end{aligned} \tag{2.5}$$

It is not hard to see that

$$\begin{aligned}&X_{k-2j-q}^{-k-1,\,k-2j}\cos\left\{(k-2j-q)M+(k-2j)g+m[\Omega-n_4(t-t_0)-\right.\\&\left.-w_{km}]+\left[m-k+2E\left(\frac{k-m}{2}\right)\right]\frac{\pi}{2}\right\}=\frac{e^{|q|}}{2}Y_{k-2j-q}^{-k-1,\,k-2j}\times\\&\times[\exp(iqg)\exp(tD)+\exp(-iqg)\exp(-tD)]=\\&=\frac{1}{2}Y_{k-2j-q}^{-k-1,\,k-2j}[\gamma^{q}\exp(iD)+\bar{\gamma}^{|q|}\exp(-iD)].\end{aligned} \tag{2.6}$$

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Substituting expression (2.6) into formula (2.1) we finally obtain

$$R = \varkappa^2 \cdot \sum_{k=2}^{\infty} \sum_{m=0}^{k} \frac{\overline{J}_{km} r_0^k}{2a^{k+1}} \cdot \sum_{j=0}^{k} \sum_{q=-\infty}^{\infty} F_{kmj}(I) \cdot Y_{k-2j-q}^{-k-1,\, k-2j}(e) \times \times \left[\gamma^{|q|} \exp(iD) + \overline{\gamma}^{|q|} \exp(-iD)\right]. \tag{2.7}$$

We will find the expression for the partial derivatives from the perturbation function according to the elements of (1.7). As is known, [4], the modified coefficients of Hansen $Y_p^{n,m}(e)$ are expanded with respect to the positive orders $e^2 = \gamma\overline{\gamma}$. Let

$$\sqrt{1-s^2}\,\frac{dF_{kmj}(I)}{ds} = \widetilde{F}_{kmj}(I), \quad \frac{dY_p^{n,m}(e^2)}{d(e^2)} = \widetilde{Y}_p^{n,m}(e^2), \tag{2.8}$$

$$\gamma^{|q|-1} = \begin{cases} \gamma^{|q|-1}, & \text{если } q \geqslant 1, \\ 0, & \text{если } q \leqslant 0, \end{cases}$$

$$\overline{\gamma}^{|q|-1} = \begin{cases} \overline{\gamma}^{|q|-1}, & \text{если } q \geqslant 1, \\ 0, & \text{если } q \leqslant 0. \end{cases} \tag{2.9}$$

Dropping the indices from the function $F_{kmj}(I)$, $\widetilde{F}_{kmj}(I)$, $Y_{k-2j-q}^{-k-1,\,k-2j}(e^2)$, $\widetilde{Y}_{k-2j-q}^{-k-1,\,k-2j}(e^2)$ and defining

$$\Sigma = \sum_{k=2}^{\infty} \sum_{m=0}^{k} \sum_{j=0}^{k} \sum_{q=-\infty}^{\infty},$$

we obtain the partial derivative of R with respect to the elements in the form

$$\begin{aligned}
\frac{\partial R}{\partial a} &= -\varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+2}} FY\left[\gamma^{|q|} \exp(iD) + \overline{\gamma}^{|q|} \exp(-iD)\right], \\
\frac{\partial R}{\partial s} &= \varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+1}\sqrt{1-s^2}} \widetilde{F}Y\left[\gamma^{|q|} \exp(iD) + \overline{\gamma}^{|q|} \exp(-iD)\right], \\
\frac{\partial R}{\partial \Omega} &= \varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+1}} miFY\left[\gamma^{|q|} \exp(iD) - \overline{\gamma}^{|q|} \exp(-iD)\right], \\
\frac{\partial R}{\partial \lambda} &= \varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+1}} (k-2j-q) iFY\left[\gamma^{|q|} \exp(iD) - \overline{\gamma}^{|q|} \exp(-iD)\right], \\
\frac{\partial R}{\partial \gamma} &= \varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+1}} F\{|q| \cdot Y\left[\gamma^{|q|-1} \exp(iD) + \gamma^{|-q|-1} \exp(-iD)\right] + \\
&\quad + \overline{\gamma}\widetilde{Y}\left[\gamma^{|q|} \exp(iD) + \overline{\gamma}^{|q|} \exp(-iD)\right]\}, \\
\frac{\partial R}{\partial \overline{\gamma}} &= \varkappa^2 \sum \frac{\overline{J}_{km} r_0^k}{2a^{k+1}} F\{|q| \cdot Y\left[\overline{\gamma}^{|-q|-1} \exp(iD) + \overline{\gamma}^{|q|-1} \exp(-iD)\right] + \\
&\quad + \gamma\widetilde{Y}\left[\gamma^{|q|} \exp(iD) + \overline{\gamma}^{|q|} \exp(-iD)\right].
\end{aligned} \tag{2.10}$$

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In equations (1.9) we neglected terms approximately equal to $J_{20}J_{km}$. For Earth satellites these terms are of the order of 10^{-9}. Since $e^2<10^{-4}$, one may also neglect terms about equal to $e^2\bar{J}_{km}$ on the right sides of the equations of perturbating motion. Thus the limits of the summation over q in formulae (2.7) and (2.10) are shortened: $-2 \le q \le 2$ in the formulae for $\frac{\partial R}{\partial \gamma}$ and $\frac{\partial R}{\partial \bar{\gamma}}$, in all other formulae it is sufficient to take $-1 \le q \le 1$.

3. THE RIGHT SIDES OF THE LAGRANGE EQUATIONS

After the substitution of formulae (2.10) into the right sides of equations (1.9), we obtain

$$\frac{da}{dt} = \varkappa\sqrt{a}\sum \frac{\bar{J}_{km} r_0^k}{a^{k+1}}(k-2j-q)iFY\left[\gamma^{[q]}\exp(iD)-\bar{\gamma}^{[q]}\exp(-iD)\right],$$

$$\frac{ds}{dt} = \frac{\varkappa\sqrt{1-s^2}}{s\sqrt{a(1-e^2)}}\sum \frac{\bar{J}_{km} r_0^k}{2a^{k+1}} iFY\{\left[\sqrt{1-s^2}(k-2j-q)-m\right]\times$$
$$\times\left[\gamma^{[q]}\exp(iD)-\bar{\gamma}^{[q]}\exp(-iD)\right]+|q|\sqrt{1-s^2}\left[(\gamma\gamma^{[q]-1}-\right.$$
$$\left.-\bar{\gamma}\,\bar{\gamma}^{[-q]-1})\exp(iD)+(\gamma\gamma^{[-q]-1}-\bar{\gamma}\,\bar{\gamma}^{[q]-1})\exp(-iD)\right]\},$$

$$\frac{d\Omega}{dt} = n_3+\frac{\varkappa\sqrt{a}}{s\sqrt{1-e^2}}\sum \frac{\bar{J}_{km} r_0^k}{2a^{k+2}}\widetilde{F}Y\left[\gamma^{[q]}\exp(iD)+\bar{\gamma}^{[q]}\exp(-iD)\right],$$

$$\frac{d\lambda}{dt} = (n_1+n_2)+\frac{\varkappa\sqrt{a}}{2}\sum \frac{\bar{J}_{km} r_0^k}{a^{k+2}}\left\{\left[2FY-\right.\right.$$
$$-\frac{\sqrt{1-s^2}}{s\sqrt{1-e^2}}\widetilde{F}Y+\frac{2\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\gamma\bar{\gamma}\widetilde{FY}\Big]\times$$
$$\times\left[\gamma^{[q]}\exp(iD)+\bar{\gamma}^{[q]}\exp(-iD)\right]+$$
$$+\frac{\sqrt{1-e^2}}{1+\sqrt{1-e^2}}FY|q|\left[(\gamma\gamma^{[q]-1}+\bar{\gamma}\,\bar{\gamma}^{[-q]-1})\exp(iD)+\right.$$
$$\left.\left.+(\gamma\gamma^{[-q]-1}+\bar{\gamma}\,\bar{\gamma}^{[q]-1})\exp(-iD)\right]\right\},$$

$$\frac{d\gamma}{dt} = n_2\cdot i\gamma+\varkappa\sqrt{a}\sum \frac{\bar{J}_{km} r_0^k}{2a^{k+2}} i\left\{\gamma\left[2\sqrt{1-e^2}F\widetilde{Y}-\right.\right.$$
$$\left.-\frac{\sqrt{1-s^2}}{s\sqrt{1-e^2}}\widetilde{F}Y\right]\left[\gamma^{[q]}\exp(iD)+\bar{\gamma}^{[q}\exp(-iD)\right]-$$
$$-\frac{\sqrt{1-e^2}}{1+\sqrt{1-e^2}}FY\gamma(k-2j-q)\left[\gamma^{[q]}\exp(iD)-\bar{\gamma}^{[q]}\exp(-iD)\right]+$$
$$\left.+2\sqrt{1-e^2}FY\cdot|q|\left[\gamma^{[-q]-1}\exp(iD)+\bar{\gamma}^{[q]-1}\exp(-iD)\right]\right\},$$

$$\frac{d\bar{\gamma}}{dt} = -n_2 i\bar{\gamma}+\varkappa\sqrt{a}\sum \frac{\bar{J}_{km} r_0^k}{2a^{k+2}} i\left\{\bar{\gamma}\left[-2\sqrt{1-e^2}F\widetilde{Y}+\right.\right.$$

$$+\frac{\sqrt{1-s^2}}{s\sqrt{1-e^2}}\tilde{F}Y\Big]\left[\gamma^{|q|}\exp(iD)+\bar{\gamma}^{|q|}\exp(-iD)\right]+$$
$$+\frac{\sqrt{1-e^2}}{1+\sqrt{1-e^2}}FY\bar{\gamma}(k-2j-q)\left[\gamma^{|q|}\exp(iD)-\bar{\gamma}^{|q|}\exp(-iD)\right]-$$
$$-2\sqrt{1-e^2}\,FY|q|\left[\gamma^{|q|-1}\exp(iD)+\bar{\gamma}^{|-q|-1}\exp(-iD)\right]\Big\}. \qquad (3.1)$$

Equations (3.1) are satisfied for all the range $0 \leqslant e < 1$. In the case of a nearly circular orbit, as was shown above, one may neglect the terms of the order of $e^2\bar{J}_{km}$ on the right sides of equations (3.1).

It is necessary for the calculation of the right sides of equations (3.1) to find the function of the inclination $F_{kmj}(I)$ and the modified Hansen coefficients $Y_{k-2j-q}^{-k-1,\,k-2j}(e^2)$, as well as $\bar{F}_{kmj}(I)$ and $\tilde{Y}_{k-2j-q}^{-k-1,\,k-2j}(e^2)$. The function $F_{kmj}(I)$ and its derivative $\bar{F}_{kmj}(I)$ may be found according to the formulae presented in the work [3] (the same notations).

Calculating the Hansen coefficients $X_p^{n,m}(e)$ and their derivatives [1,5], we find

$$Y_p^{n,\,m} = e^{-|m-p|}X_p^{n,\,m},$$
$$\tilde{Y}_p^{n,\,m} = -\frac{|m-p|}{2}e^{-|m-p|-2}X_p^{n,\,m} + \frac{e^{-|m-p|-1}}{2}\frac{dX_p^{n,\,m}}{de}. \qquad (3.2)$$

The calculation of the coefficients $Y_p^{n,m}(e^2)$ and the corresponding derivatives may also be carried out according to the formulae

$$Y_p^{n,\,m}(e^2) = \sum_{l=0}^{\infty}\prod_{p-m}^{|m-p|+2l}(n,\,m)\,e^{2l},$$
$$\tilde{Y}_p^{n,\,m}(e^2) = \sum_{l=1}^{\infty} l\prod_{p-m}^{|m-p|+2l}(n,\,m)\,e^{2l-2}, \qquad (3.3)$$

where $\prod_{p-m}^{|m-p|+2l}(n,\,m)$ is the Newcome polynomial. The formulae (3.3) are easily obtained from formulae (2.3) and Equation (5) presented in the article of B. A. Brumberg [6]. Methods of calculation of the Newcome polynomials may be found in articles [5, 7, 8].

The modified Hansen coefficients impose important properties

$$Y_p^{n,\,m} = Y_{-p}^{n,\,-m}. \qquad (3.4)$$

If p=0, then the coefficients $Y_0^{n,m}$ lead to the hyper-geometric /170
polynomials

$$Y_0^{n,m} = \frac{(-1)^{|m|}(n+2)_{|m|}}{(1)_{|m|}} \frac{(1+\beta^2)^{-n-1}}{(1+\sqrt{1-e^2})^{|m|}} \times$$
$$\times F(-n-1, -n+|m|-1, 1+|m|; \beta^2), \ldots, n \geqslant -1,$$
$$Y_0^{n,m} = \frac{(-1)^{|m|}(n+2)_{|m|}}{(1)_{|m|}2^{|m|}}(1-e^2)^{n+\frac{3}{2}} \times$$
$$\times F\left(\frac{n+|m|+2}{2}, \frac{n+|m|+3}{2}, 1+|m|; e^2\right), \ldots, n < -1. \qquad (3.5)$$

Here, for n = -2, -3, ..., - m -1 the coefficient $Y_0^{n,m}(e^2)=0$.
In formulae (3.5) assumed are the definitions

$$\beta = \frac{e}{1+\sqrt{1-e^2}},$$
$$(\alpha)_0 = 1, (\alpha)_1 = \alpha, (\alpha)_2 = \alpha(\alpha+1), \ldots, (\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1). \qquad (3.6)$$

It is obvious, if $e < 0.01$, that it is sufficient to find $Y_p^{n,m}(e^2)$ to an accuracy of e^2. In this case the calculation of the coefficients $Y_p^{n,m}(e^2)$ and their derivatives introduced into the right sides of (3.1) may be carried out according to the formulae

$$Y_0^{n,m} = \frac{(-1)^{|m|}(n+2)_{|m|}}{(1)_{|m|}2^{|m|}},$$
$$\widetilde{Y}_0^{n,m} \frac{(-1)^{|m|}(n+2)_{|m|}}{(1)_{|m|}2^{|m|}}\left[\frac{(n+|m|+2)(n+|m|+3)-2(2n+3)(1+|m|)}{4(1+|m|)}\right], n<-1,$$

and in the general case

$$Y_p^{n,m} = \prod_{p-m}^{|m-p|}(n, m), \quad \widetilde{Y}_p^{n,m} = \prod_{p-m}^{|m-p|+2}(n, m). \qquad (3.7)$$

We will be solving the system of equations (3.1) by a method of successive approximations. For the zeroth approximation we will take the unperturbed Eulerian motion

$$a = a_0, \; s = s_0, \; e = e_0,$$
$$\lambda = (n_1+n_2)t+\lambda_0, \; \Omega = n_3 t + \Omega_0, \qquad (3.8)$$

where n_1, n_2, n_3 are defined through a_0, e_0, s_0 according to formulae (1.3). As concerns γ, $\bar{\gamma}$, it is necessary before all else to determine their secular change.

4. THE SECULAR MOVEMENT OF THE LAPLACE VECTOR

For obtaining the secular movement of the Laplace vector γ and the conjugate magnitude $\bar{\gamma}$ we will leave on the right side of the last two equations of (3.1) only the terms not clearly depending on t and λ. We set

$$m=0,\ k-2j-q=0. \tag{4.1}$$

Then /171

$$D=-\frac{k\pi}{2}+E\left(\frac{k}{2}\right)\pi.$$

We keep in mind the following equations [1, 3]:

$$\begin{aligned} Y_0^{-k-1,\pm k}&=0,\quad Y_p^{n,m}=Y_{-p}^{n,-m},\\ F_{k,0,j}(I)&=(-1)^k F_{k,0,k-j}(I). \end{aligned} \tag{4.2}$$

We will keep only the terms which are linear with respect to γ, $\bar{\gamma}$, since we will consider e < 0.01. It is easy to see that the positive harmonics produce only terms proportional to γ, $\bar{\gamma}$, multiplied by purely imaginary constants, but the negative harmonics give only a real constant. Summing all the harmonics, we obtain

$$\begin{aligned} \frac{d\gamma}{dt}&=iE_1\gamma+iE_2\bar{\gamma}+E_3,\\ \frac{d\bar{\gamma}}{dt}&=-iE_2\gamma-iE_1\bar{\gamma}+E_3, \end{aligned} \tag{4.3}$$

where

$$\begin{aligned} E_1&=n_2+\sum_{k=2}^{\infty}\frac{\varkappa\sqrt{a}\,\bar{J}_{2k,0}r_0^{2k}}{a^{2k+2}}\left[2\sqrt{1-e^2}\,F_{2k,0,k}(I)\,\widetilde{Y}_0^{-2k-1,0}-\right.\\ &\left.-\frac{\sqrt{1-s^2}}{s\sqrt{1-e^2}}\widetilde{F}_{2k,0,k}(I)\,Y_0^{-2k-1,0}\right],\\ E_2&=4\varkappa\sqrt{a(1-e^2)}\sum_{k=2}^{\infty}\frac{\bar{J}_{2k,0}r_0^{2k}}{a^{2k+2}}F_{2k,0,k-1}(I)\,Y_0^{-2k-1,2},\\ E_3&=2\varkappa\sqrt{a(1-e^2)}\sum_{k=2}^{\infty}\frac{-\bar{J}_{2k-1,0}r_0^{2k-1}}{a^{2k+1}}F_{2k-1,0,k-1}(I)\,Y_0^{-2k,1}. \end{aligned} \tag{4.4}$$

Thus, for the calculation of the secular movement of γ, $\bar{\gamma}$ we have a nonhomogeneous linear system (4.3) with constant coefficients (4.4). The roots of the characteristic equation are equal to $\pm\sigma$, where

$$\sigma = \sqrt{E_1^2 - E_2^2}\,\operatorname{sign} E_1. \tag{4.5}$$

From formulae (4.4) and (1.3), (1.4) it is seen that

$$E_1 \sim n_2,\quad E_2 \sim J_{2h,0}.$$

We will assume that $E_1^2 - E_2^2 > 0$. This last inequality is always satisfied outside the neighborhood of critical inclination $4-5s^2 = 0$, $I = 63°26'$. For the Earth, as an example, it is sufficient to exclude the neighborhood of latitude of 20'.

The general solution of the system (4.3) is /172

$$\gamma = \frac{iE_3}{E_1 - E_2} + K_1(E_1+\sigma)\exp(i\sigma t) - K_2E_2\exp(-i\sigma t),$$
$$\bar{\gamma} = -\frac{iE_3}{E_1 - E_2} - K_1E_2\exp(i\sigma t) + K_2(E_1+\sigma)\exp(-i\sigma t),$$

where K_1 and K_2 are arbitrary constants of integration. Since γ, $\bar{\gamma}$ are conjugate, K_1 and K_2 are also conjugate. We will set

$$K_1 = K\exp(-i\sigma\tau),\quad K_2 = K\exp(i\sigma\tau).$$

Here K and τ are real arbitrary constants. Then the general solution of system (4.3) takes the form

$$\gamma = \frac{iE_3}{E_1 - E_2} + K(E_1+\sigma)\exp i\sigma(t-\tau) - KE_2\exp i\sigma(\tau - t),$$
$$\bar{\gamma} = -\frac{iE_3}{E_1 - E_2} - KE_2\exp i\sigma(t-\tau) + K(E_1+\sigma)\exp i\sigma(\tau - t). \tag{4.6}$$

Separating the real and imaginary parts, we obtain

$$\beta_1 = K(E_1 - E_2 + \sigma)\cos\sigma(t-\tau),$$
$$\beta_2 = K(E_1 + E_2 + \sigma)\sin\sigma(t-\tau) + \frac{E_3}{E_1 - E_2}. \tag{4.7}$$

Thus, a hodograph of the Laplacian vector in secular motion describes an ellipse of very small eccentricity with its center at the point $(0, \frac{E_3}{E_1-E_2})$:

$$\frac{\beta_1^2}{K^2(E_1-E_2+\sigma)^2}+\frac{\left(\beta_2-\frac{E_3}{E_1-E_2}\right)^2}{K^2(E_1+E_2+\sigma)^2}=1.$$

For K=0, we observe a state of rest (e=const $\neq$0, g = $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ depending on the sign of $E_3/(E_1-E_2)$.

For

$$|K|<\left|\frac{E_3}{(E_1-E_2)(E_1+E_2+\sigma)}\right|$$

the ellipse does not intersect the straight line β_1=0 (e $\geq$ const > >0, g is a periodic function of time).

For

$$|K|=\left|\frac{E_3}{(E_1-E_2)(E_1+E_2+\sigma)}\right|$$

the ellipse touches the line β_1=0 (e periodically returns to zero and at these moments g undergoes a jump of π).

For

$$|K|>\left|\frac{E_3}{(E_1-E_2)(E_1+E_2+\sigma)}\right|$$

the ellipse contains the origin of the coordinates inside itself (e $\geq$ const $\geq$0, g has a secular motion).

If E_3=0 (in particular, for the omission of the negative harmonics), the picture is simplified: for K=0, we have a circular motion; for K$\neq$0 the ellipse contains within itself the origin of the coordinates (e $\geq$ const > 0, g has a secular motion). /173

The following are the basic qualitative conclusions.

1. If at the initial moment the value of the eccentricity is of the order of 10^{-2} to 10^{-3}, then the order of the eccentricity will not change:

$$e \sim e_o \sim 10^{-2} \text{ to } 10^{-3}.$$

2. If the initial value of the eccentricity is of the order of 10^{-3} and smaller, down to zero, then for the most part the value of the eccentricity $e_{max} \sim 10^{-3}$ will be reached in the extent of time of $\frac{2\pi}{\sigma}$.

5. FULL MOTION OF THE LAPLACIAN VECTOR

For obtaining the full motion of the Laplacian vector we take, on the right side of the last two equations of (3.1), a, s, and e as constants; λ and Ω as linear functions of time; γ and $\bar{\gamma}$ as solutions (4.6) of the simplified equations (4.3). Calculating the integral from the terms not depending on γ and Ω we again obtain the expression (4.6). The remaining terms are proportional to $\gamma^k \bar{\gamma}^q \exp(i\bar{n}t)$, while $\bar{n} \sim 1$. Integrating by parts, we find

$$\int \gamma^k \bar{\gamma}^{|q|} \exp(i\tilde{n}t)\, dt = \frac{\gamma^k \bar{\gamma}^{|q|}}{i\tilde{n}} \exp(i\tilde{n}t) - \frac{1}{i\tilde{n}} \int \left(k\gamma^{k-1}\bar{\gamma}^q \frac{d\gamma}{dt} + q\gamma^k \bar{\gamma}^{q-1} \frac{d\bar{\gamma}}{dt} \right) \exp(i\tilde{n}t)\, dt.$$

The unintegrated term is small (for an AES it is of the order of 10^{-3}), since the derivatives $\frac{d\gamma}{dt}$ and $\frac{d\bar{\gamma}}{dt}$ are proportional to J_{20}. With accepted accuracy, one may consider

$$\int \gamma^k \bar{\gamma}^q \exp(i\tilde{n}t)\, dt = \gamma^k \bar{\gamma}^q \int \exp(i\tilde{n}t)\, dt. \qquad (5.1)$$

The final expressions for the full motion of the Laplacian vector will be derived below.

6. THE SECULAR AND LONG-PERIOD PERTURBATIONS, CAUSED BY ZONAL HARMONICS

The secular and long-period perturbations of γ, $\bar{\gamma}$ were obtained in paragraph 4. For the finding of the secular and long-period perturbations in the elements a, s, Ω, and λ, we will return to the questions of (1.9). We set

$$k-2j-q=0, \quad m=0, \tag{6.1}$$

so that the right side of the Lagrange equations will not depend on time in unperturbed motion. As is seen from formulae (2.10),

$$\left[\frac{dR}{d\lambda}\right]=0, \quad \left[\frac{dR}{d\Omega}\right]=0,$$

consequently, $\left[\frac{da}{dt}\right] = 0$, i.e., the secular and long-period perturbations are omitted on the large semi-axis. 174

The integration of the remaining equations is carried out under the assumptions that on the right side $a=a_0$, $s=s_0$, but γ, $\bar{\gamma}$ are defined by equations (4.6) or β_1 and β_2, by the equations (4.7). Neglecting the terms of the order $e^2\bar{J}_{k0}$, we obtain from formulae (2.10) and (4.2):

$$\begin{aligned}\left[\frac{\partial R}{\partial a}\right]&=\varkappa^2\sum_{k=2}^{\infty}\left[\frac{\bar{J}_{2k,0}r_0^{2k}}{a^{2k+2}}(2k+1)F_{2k,0,k}Y_0^{-2k-1,0}+\right.\\&\left.+\frac{4\bar{J}_{2k-1,0}r_0^{2k-1}}{a^{2k+1}}k\beta_2F_{2k-1,0,k-1}Y_0^{-2k,1}\right],\\\left[\frac{\partial R}{\partial s}\right]&=\varkappa^2\sum_{k=2}^{\infty}\left[\frac{\bar{J}_{2k,0}r_0^{2k}}{a^{2k+1}\sqrt{1-s^2}}\widetilde{F}_{2k,0,k}Y_0^{-2k-1,0}+\right.\\&\left.+\frac{2\bar{J}_{2k-1,0}r_0^{2k-1}}{a^{2k}\sqrt{1-s^2}}\beta_2\widetilde{F}_{2k-1,0,k-1}Y_0^{-2k,1}\right].\end{aligned} \tag{6.2}$$

Considering only the secular motion of γ and $\bar{\gamma}$, we obtain from the last two equations of (1.9)

$$\begin{aligned}t\left(\gamma\frac{\partial R}{\partial\gamma}-\bar{\gamma}\frac{\partial R}{\partial\bar{\gamma}}\right)&=-\varkappa\sqrt{a}\,E_3\beta_1,\\\gamma\frac{\partial R}{\partial\gamma}+\bar{\gamma}\frac{\partial R}{\partial\bar{\gamma}}&=-\varkappa\sqrt{a}\,E_3\beta_2.\end{aligned} \tag{6.3}$$

Substituting the equations (4.7), (6.2) and (6.3) into the right side of the equations of motion and integrating, we find

$$
\begin{aligned}
[s] &= \frac{1-s^2}{s}\left(1+\sqrt{\frac{E_1-E_2}{E_1+E_2}}\right)KE_3\sin\sigma(t-\tau),\\
[\Omega] &= \left[n_3+\frac{\varkappa\sqrt{a}}{s}\sum_{k=2}^{\infty}\frac{\bar{J}_{2k,0}r_0^{2k}}{a^{2k+2}}\tilde{F}_{2k,0,k}Y_0^{-2k-1,0}\right]t-\\
&-\frac{2\varkappa\sqrt{a}}{s}K\left(1+\sqrt{\frac{E_1+E_2}{E_1-E_2}}\right)\sum_{k=2}^{\infty}\frac{\bar{J}_{2k-1,0}r_0^{2k-1}}{a^{2k+1}}\times\\
&\times\tilde{F}_{2k-1,0,k-1}Y_0^{-2k,1}\cos\sigma(t-\tau),\\
[\lambda] &= \left\{(n_1+n_2)-\varkappa\sqrt{a}\sum_{k=2}^{\infty}\frac{\bar{J}_{2k,0}r_0^{2k}}{a^{2k+2}}Y_0^{-2k-1,0}\times\right.\\
&\left.\times\left[2(2k+1)F_{2k,0,k}+\frac{\sqrt{1-s^2}}{s}\tilde{F}_{2k,0,k}\right]\right\}t+\\
&+K\left(1+\sqrt{\frac{E_1+E_2}{E_1-E_2}}\right)\left\{E_3+2\varkappa\sqrt{a}\sum_{k=2}^{\infty}\frac{\bar{J}_{2k-1,0}r_0^{2k-1}}{a^{2k+1}}Y_0^{-2k,1}\times\right.\\
&\left.\times\left[4kF_{2k-1,0,k-1}+\frac{\sqrt{1-s^2}}{s}\tilde{F}_{2k-1,0,k-1}\right]\right\}\cos\sigma(t-\tau).
\end{aligned}
\tag{6.4}
$$

We note that the results of this paragraph may be obtained directly from equation (3.1).

7. SHORT-PERIOD PERTURBATIONS

For the calculation of the short-period terms, it is necessary to integrate the system of equations (3.1) under the assumptions that k-2j-q and m and not simultaneously equal to zero. As before, we will keep on the right sides of equations (3.1) only the terms linear with respect to γ and $\bar{\gamma}$.

During the integration we will take

$$a=a_0,\ s=s_0,\ \lambda=\bar{n}_1t+\lambda_0,\ \Omega=\bar{n}_3t+\Omega_0,$$

where $\bar{n}_1$ and $\bar{n}_3$ are coefficients for t in the corresponding equations of (6.4).

During the computation of the short-period terms, just as in paragraph 5, it is sufficient to integrate only exp(iD), taking the coefficient for the exponent out of the integral sign.

After the integration, it is best to consider the elements on the right sides jointly with the secular and long-period parts.

We note that for the perturbation of the average argument of the latitude it is necessary to add

$$-\frac{3}{2}\frac{n_1}{a}\int\{a\}\,dt.$$

Designating the short-period terms by braces, we finally obtain

$$\begin{aligned}
\{a\} &= \varkappa\sqrt{a}\sum\frac{\overline{J}_{km}r_0^k}{a^{k+1}}\cdot\frac{(k-2j-q)FY}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}\times\\
&\quad\times\left[\gamma^{|q|}\exp(iD)+\overline{\gamma}^{|q|}\exp(-iD)\right],\\
\{s\} &= \frac{\varkappa\sqrt{1-s^2}}{s\sqrt{a(1-e^2)}}\sum\frac{\overline{J}_{km}r_0^k}{2a^{k+1}}\cdot\frac{FY}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}\times\\
&\quad\times\{\left[\sqrt{1-s^2}(k-2j-q)-m\right]\left[\gamma^{|q|}\exp(iD)+\overline{\gamma}^{|q|}\exp(-iD)\right]+\\
&\quad+|q|\left[(\gamma\gamma^{|q|-1}-\overline{\gamma}\,\overline{\gamma}^{|-q|-1})\exp(iD)-\right.\\
&\quad\left.-(\gamma\gamma^{|-q|-1}-\overline{\gamma}\,\overline{\gamma}^{|q|-1})\exp(-iD)\right]\sqrt{1-s^2}\},\\
\{\Omega\} &= \frac{\varkappa\sqrt{a}}{is}\sum\frac{\overline{J}_{km}r_0^k}{2a^{k+2}}\cdot\frac{FY}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}\times\\
&\quad\times\left[\gamma^{|q|}\exp(iD)-\overline{\gamma}^{|q|}\exp(-iD)\right],\\
\{\gamma\} &= \varkappa\sqrt{a}\sum\frac{\overline{J}_{km}r_0^k}{2a^{k+2}}\cdot\frac{i}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}\times\\
&\quad\times\left\{\gamma\left[2F\widetilde{Y}-\frac{\sqrt{1-s^2}}{s}\widetilde{F}Y\right]\left[\gamma^{|q|}\exp(iD)-\overline{\gamma}^{|q|}\exp(-iD)\right]-\right.\\
&\quad-\frac{\gamma}{2}(k-2j-q)FY\left[\gamma^{|q|}\exp(iD)+\overline{\gamma}^{|q|}\exp(-iD)\right]+\\
&\quad\left.+2|q|FY\left[\gamma^{|-q|-1}\exp(iD)-\overline{\gamma}^{|q|-1}\exp(-iD)\right]\right\},\\
\{\lambda\} &= \frac{\varkappa\sqrt{a}}{2i}\sum\frac{\overline{J}_{km}r_0^k}{a^{k+2}}\cdot\frac{1}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}\times\\
&\quad\times\left\{\left[2FY-\frac{3n_1(k-2j-q)}{\widetilde{n}_1(k-2j-q)+m(\widetilde{n}_3-n_4)}FY-\right.\right.\\
&\quad\left.-\frac{\sqrt{1-s^2}}{s}\widetilde{F}Y+\gamma\overline{\gamma}F\widetilde{Y}\right]\left[\gamma^{|q|}\exp(iD)-\overline{\gamma}^{|q|}\exp(-iD)\right]+\\
&\quad+\frac{|q|}{2}FY\left[(\gamma\gamma^{|q|-1}+\overline{\gamma}\,\overline{\gamma}^{|-q|-1})\exp(iD)-\right.\\
&\quad\left.\left.-(\gamma\gamma^{|-q|-1}+\overline{\gamma}\,\overline{\gamma}^{|q|-1})\exp(-iD)\right]\right\}.
\end{aligned}\tag{7.1}$$

Here the indices and limits of summation are the same as in formula (3.1), but the terms, for which k-2j-q-m=0, must be omitted.

In formulae (7.1), one may switch to real variables. However, the complex form is preferable. In the first place, the

coefficients for cosines and sines depend on the index q in an especially complicated manner in contrast to the coefficients of the exponents. In the second place, in the real case, the representation is larger by one formula. Besides this, the equating of the imaginary part of exp(iD), ... to zero serves as a good control of the calculation.

Thus, the perturbations of a nearly-circular orbit are obtained. In conclusion, we turn attention to one important case. As was shown in paragraph 4, an eccentricity, small at the initial moment, remains small in the following time. Therefore, it is sufficient to determine once what formulae are to be used: those obtained here for near-circular orbits or those obtained in [1] for elliptical orbits.

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